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A non optimization-based method for reconstructing wind instruments bore shape

G. Le Vey

IRCCyN/Ecole des Mines de Nantes, 4, rue A. Kastler, BP 20722, 44307 Nantes, France
levey@emn.fr

The purpose of this presentation is twofold : first, to bring attention to an existing "direct" *layer-peeling* (or *layer-stripping*) method for bore reconstruction of wind instruments as an alternative to the conventional one in this field of research. Second, to relate the above to the schwarzian derivative for the corresponding Sturm-Liouville equation. It could complement the presented algorithm and be useful for the purpose of analysis. These points are discussed theoretically but the question of practical measurements (input impulse response or input impedance) is beyond the scope of this work. One contemplated application is to the study of ancient wind instruments, with a goal of better precision in bore reconstruction.

1 Introduction

The idea from which originated this work was to try to use a simple property of what is known as the *schwarzian derivative* -or *schwarzian* for short- of a function (see section 3), in the context of bore reconstruction of wind instruments. It has relationship with the Sturm-Liouville problem for second-order differential equations such as the one appearing in wave propagation in a one dimensional finite medium. In brief, the schwarzian relates the potential (in the sense of a Schrödinger-type formulation) of such equations to the quotient of two independent solutions of the problem. Once this observation made, the question of how to get these two solutions remains. There are two possibilities : either one has access to measured ones or one is able to estimate them in an indirect manner from other measurements (usually at the boundary). This lead us to have a look at methods of inverse scattering for one-dimensional layered media, as applied to the problem of bore reconstruction. Thus, in the first part (section 2) is exposed one layer-peeling algorithm [8, 7], a feature of which is to make no use of the reflection coefficient to compute the unknown varying diameter of a pipe but proceeds in a direct way for this task. It does not seem to have been used in the present context. The second part (section 3) of this work exposes the results that we obtained for the bore reconstruction problem while investigating the schwarzian derivative. It is fair to say that at the present stage, this part is rather of theoretical interest but we think that it could be useful for further analysis.

Most of the methods for bore reconstruction that are not optimization-based, now call for a so-called *layer-peeling* algorithm([7, 1, 2] and references therein) that originated from inverse scattering in seismology. It uses a local analysis of wave propagation that leads to recursive methods of dynamic deconvolution of signals for identifying layered media. The central idea is the *possibility to determine the nearest scattering layer from a causal input-output pair recorded at its [the medium] left boundary* [5]. These algorithms have proven to be more efficient than earlier methods ([19] ; see e.g. [7] for more references) based on an integral representation which originated in the work of Gelfand-Levitan [14].

The first purpose of this presentation is to bring attention to a direct layer-peeling algorithm, in its continuous, differential setting, that does not proceed through the usual intermediate step of estimating the local reflection coefficients, from which the radius of a variable section tubular acoustic waveguide is deduced. Although this last approach is commonly used either with time-domain measurements [18, 9] or frequency-domain ones [12], it is a fact that it can be sensitive to discontinuities [18, 12], even in the derivative of the bore section, because, in the continuous setting, the reflection coefficient is expressed as a function of the first derivative of the local impedance function [7], requiring thus the differentiability of this last one. The presented method could give better

results but this has to be confirmed practically. One simple difference is that an input with regularity not less than that of a step function is required instead of an impulse. Notice also that in the context of bore reconstruction for acoustical pipes, the derivation of a layer-peeling algorithm frequently assumes that the pipe is made of several cylinders joined together¹. This assumption is not made in the presented algorithm, as its derivation, in its continuous version, starts from the propagation PDEs.

The method on which we bring attention was exposed in [7] in a system theoretic fashion, or [8] directly on the field equations. The results shown in [1, 2] also rely upon algorithms derived in a general context in [7] while including losses in the waveguide. Nevertheless, observe that [7] also treats some situations with losses and [6] considers the case with noisy data while in [5] a detailed review of layer-peeling algorithms in the discrete situation is presented. The presented direct method relies upon knowing pressure and volume velocity at the entrance of a wave guide. The basic idea behind time domain measurements is separation of forward and backward propagating pressure waves. Measuring these two entities separately is entirely equivalent to measuring pressure and volume velocity [2], as used for frequency-domain methods [11]. Two different experimental approaches for getting scattering data are either pulse reflectometry-based methods [18, 9], i.e. time-domain ones, or input impedance-based, i.e. frequency-domain ones [3, 11, 12]. At least theoretically, they are equivalent, being a Fourier pair. It is not our purpose here to discuss the respective pros and cons of each method in real experiments (see [12, 18, 9]).

2 A direct layer-peeling algorithm

Consider the following first-order system of PDEs :

$$\begin{cases} \frac{\partial p}{\partial x} + \frac{\rho}{A} \frac{\partial u}{\partial t} &= 0 \\ \frac{\partial u}{\partial x} + \frac{A}{\rho c^2} \frac{\partial p}{\partial t} &= 0 \end{cases} \quad (1)$$

that models the plane wave propagation inside a one-dimensional acoustic wave guide [10] ("telegrapher's equations"). In these equations, $p(x, t)$ is the pressure, $u(x, t)$ the volume velocity, ρ the gas density, c the velocity of sound, $A(x)$ the cross-section area of the tube at abscissa x . A change of variable allows to symmetrize this system : consider the travel time τ such that $c\tau = x$ (c is assumed to be constant in the medium but this assumption is not necessary, see [8]). The above PDE system can then be written equivalently as :

$$\begin{cases} \frac{\partial p}{\partial \tau} + \frac{c\rho}{A} \frac{\partial u}{\partial t} &= 0 \\ \frac{\partial u}{\partial \tau} + \frac{A}{c\rho} \frac{\partial p}{\partial t} &= 0 \end{cases} \quad (2)$$

¹[1] investigates also the case of conical sections

or, in matrix notation :

$$\begin{pmatrix} p \\ u \end{pmatrix}_\tau + \begin{pmatrix} 0 & \zeta \\ \zeta^{-1} & 0 \end{pmatrix} \begin{pmatrix} p \\ u \end{pmatrix}_t = 0, \quad (3)$$

with $\zeta = \frac{c\rho}{A}$ and indices stand for derivatives with respect to τ and t . As the medium is assumed at rest before $t = 0$, one has $p(x, 0) = u(x, 0) = 0, t < 0$. One has also boundary conditions $p(0, t)$ and $u(0, t)$ for which the Cauchy problem for (3) is well-posed when they are given on $t = 0$ or $x = 0$. Actually in the present context, ζ is unknown, usually $p(0, t)$ is imposed (e.g. as an impulse or a step function) and $u(0, t)$ is a measured quantity.

In order to derive the algorithm [7], some changes of variables are useful. Usually, the state variables p, u are normalized using ζ as :

$$\begin{cases} P(\tau, t) &= p(\tau, t)\zeta(\tau)^{-1/2} \\ U(\tau, t) &= u(\tau, t)\zeta(\tau)^{1/2} \end{cases} \quad (4)$$

while defining the local reflexion coefficient :

$$k(\tau) = \zeta(\tau)^{-1/2} \frac{d}{d\tau} (\zeta(\tau)^{1/2}) = \frac{1}{2} \frac{d}{d\tau} (\ln \zeta(\tau)) \quad (5)$$

in order to get the equivalent system :

$$\begin{cases} P_\tau + U_t + kP &= 0 \\ U_\tau + P_t - kU &= 0 \end{cases} \quad (6)$$

Notice that the following relation holds :

$$\frac{p(\tau, t)}{u(\tau, t)} = \frac{P(\tau, t)}{U(\tau, t)} \zeta(\tau) \quad (7)$$

Then, the usual forward (F) and backward (B) propagating waves are obtained as respectively :

$$\begin{cases} F(\tau, t) &= \frac{1}{2}(P(\tau, t) + U(\tau, t)) \\ B(\tau, t) &= \frac{1}{2}(P(\tau, t) - U(\tau, t)) \end{cases} \quad (8)$$

For this new variables, the first-order PDE system becomes :

$$\begin{cases} F_\tau + F_t + kB &= 0 \\ B_\tau - B_t + kF &= 0 \end{cases} \quad (9)$$

This set of PDEs is a possible starting point for the derivation of layer-peeling algorithms used for bore reconstruction of wind instruments, ordinarily in a discrete setting. It explicitly makes the reflection coefficient k appear, due to the fact that a leading impulse is present in the signal used to probe the medium. Thus, looking at equation (5), one sees that differentiability of ζ is required. When passing to the discrete situation, discontinuities in the derivative of ζ are likely to pose problems.

The inverse scattering problem is then as follows : assuming the medium is quiescent at $t = 0$, a known probing waveform $F(\tau, t)$ propagates from $x = 0$ to the right, starting at $t = 0$. This waveform is usually an impulse followed by a

piecewise continuous function. The measured data is the left propagated waveform, recorded at $x = 0, B(0, t)$. With this data, the problem is to reconstruct $\zeta(\tau)$. Obviously, one cannot expect more than getting ζ as a function of τ . But considering that one can estimate c and ρ , assumed to be constant in the pipe, A can also be obtained as a function of x . The algorithm will produce, given the above data at, say, the left narrow end of the pipe², an approximation of the local reflection coefficient, from which an estimate of the local section $A(x)$ at x can be computed [1] : in the discrete situation, relation (5) translates to : $k = (D_2 - D_1)/(D_2 + D_1)$ where D_2, D_1 are respectively the diameter apart a junction of two adjacent cylindrical sections of the pipe.

But there is another slightly different way to proceed : assume that the probing wave $F(\tau, t)$ *does not contain* a leading impulse and is a piecewise continuous function starting at $t = 0$ (e.g. a unit step function). By causality, one has [7] :

$$\begin{cases} F(\tau, t) &= f(\tau, t)Y(t - \tau) \\ B(\tau, t) &= b(\tau, t)Y(t - \tau) \end{cases} \quad (10)$$

where Y is the Heaviside step function and f, b are piecewise continuous functions. We can use then the method of propagation of singularities [8] : substituting from (10) into equations (9) and identifying coefficients of functions having equal regularity, one gets : $b(\tau, \tau^+) = 0$ for $\tau^+ > \tau$. Thanks to the second equation (8) one gets : $P(\tau, \tau^+) = U(\tau, \tau^+)$ and thus, from (4) :

$$\frac{p(\tau, \tau^+)}{u(\tau, \tau^+)} = \zeta(\tau) \quad (11)$$

Then, starting from $\tau = 0$ where $\zeta(0)$ can be measured and $p(0, t), u(0, t)$ are measured data too, one can successively compute $p(\tau, t)$ and $u(\tau, t)$ from the left end, using equality (11) and equations (3) successively in a recursive way. Hence, up to direct estimation of c and ρ , one can directly reconstruct $A(x)$ in an recursive manner, without the intermediate calculation of the reflection coefficient. Another equivalent derivation of the above, directly from the field equations (3) can be found in [8] where a discrete version of this algorithm, using the method of characteristics, is also given and analyzed. It has to be noticed that this algorithm reconstructs not only ζ (i.e. A) but also the solution of (3) inside the bore : this will be useful for the analysis in the next section. Observe that, as input impedance measurements devices make use of pressure and volume velocity measurements, they could be used in conjunction with the above method : one just has to “plug” the corresponding algorithm to an existing set-up.

²see [12] for observations about increased accuracy when measurements are made from the large side of a horn

3 Schwarzian of the horn equation

As mentionned in the introduction, the subject of this section was the starting point of the presented work. Due to its very nature, it nevertheless lead to a detour through layer-peeling algorithms such as presented above. The reason is that for using the schwarzian derivative [16], one has to know by some means -measuring or computing- the quotient of two independent solutions of a Sturm-Liouville equation such as the one that can appear in the wave horn equation. It appears then that the following is, at the present stage, rather theoretical, even if the schwarzian can be computed in the course of the previous layer-peeling algorithm, as it will be shown. Consider the wave equation for the pressure p inside an axially symmetric pipe of length L , with variable cross-section $A(x) = \pi D^2(x)/4$, $x \in [0, L]$:

$$\frac{1}{c^2} p_{tt} - p_{xx} - 2 \frac{D_x}{D} p_x = 0 \quad (12)$$

Where $p(x)$ is the pressure³. Applying the Fourier transformation in time, one gets :

$$\hat{p}_{xx} + 2 \frac{D_x}{D} \hat{p}_x + \frac{\omega^2}{c^2} \hat{p} = 0 \quad (13)$$

whith $\hat{p}(x, \omega)$ the Fourier transform of $p(x, t)$. Equation (13) is a particular case of the general Sturm-Liouville second-order differential equation :

$$y_{xx}(x) + q(x)y_x(x) + p(x)y(x) = 0 \quad (14)$$

According to the theory of this class of equations, the space of solutions is two-dimensional vector space on the constants, spanned by any two linearly independent solutions, say y_1, y_2 . Consider now the so-called *schwarzian derivative* [16] (or *schwarzian*) of a sufficiently regular function f , defined as :

$$S(f(x)) = \frac{f_{xxx}(x)}{f_x(x)} - \frac{3}{2} \left(\frac{f_{xx}(x)}{f_x(x)} \right)^2 \quad (15)$$

It has the following property : assume the ratio of two solutions of (14), $f = \frac{y_1}{y_2}$, is known. Then it can be proven that :

$$S(f) = 2p - \frac{1}{2}q^2 - q_x \quad (16)$$

This does not depend on the particular choice for y_1, y_2 , as $S(f)$ is invariant under homographic transformations. Computing expression (16) for the actual situation described by equation (13), one gets after straightforward calculations :

$$\frac{1}{2} S\left(\frac{\hat{p}_1}{\hat{p}_2}\right) = \frac{\omega^2}{c^2} - \frac{D_{xx}}{D} \quad (17)$$

from which the following second-order linear differential equation for D is obtained :

$$D_{xx}(x) = \left(\frac{\omega^2}{c^2} - \frac{1}{2} S\left(\frac{\hat{p}_1}{\hat{p}_2}\right)(x, \omega) \right) D(x) \quad (18)$$

³ D is chosen instead of A as unknown parameter in order to obtain a simpler expression for the schwarzian

Now, observe that $D(0)$ can be known by measurement and $D_x(0)$ can be estimated as well. In the case of usual wind instruments, the first part of the pipe is usually close to a cylinder so that $D_x(0) \approx 0$. As (18) is a second order linear differential equation with variable coefficient defined on $[0, L]$, it can theoretically be solved. One possible use of this equation is for theoretical analysis of the results coming from the layer-peeling algorithms, that will be the subject of future work. Remember also that the so-called *inverse method of design* [13] proceeds using a direct inversion of (13) :

$$\frac{d \ln A}{dx} = 2 \frac{D_x}{D} = - \frac{\hat{p}_{1,xx} + \frac{\omega^2}{c^2} \hat{p}_1}{\hat{p}_{1,x}} \quad (19)$$

for one given solution and given boundary conditions. Hence, equations (18), (19) must be compatible, which can be checked. Notice that both equations separate the geometry of the bore (D and its derivatives) from what concerns solutions (\hat{p}_1, \hat{p}_2). One interest of (18) is that it makes appear the *horn function* D''/D that contains complete information about the shape of the horn [4]. One simple application of (18) can be for smoothing the cross-section $A(x)$ estimated through the layer-peeling algorithm of section 2.

3.1 On the computation of the schwarzian

One details how to compute the schwarzian, from the only knowledge of the causal input-output scattering data for the layer-peeling algorithm of section 2. It has been seen that the outputs of this algorithm are $p_1(\tau, t)$ and $u_1(\tau, t)$ ⁴ the Fourier transform of which are $\hat{p}_1(\tau, \omega)$ and $\hat{u}_1(\tau, \omega)$, together with $\zeta(\tau)$. Remembering the results of [13] ([15] gives the general result for equation (14)), one knows that from \hat{p}_1 a second solution \hat{p}_2 can be obtained as :

$$\hat{p}_2(\tau, \omega) = \hat{p}_1(\tau, \omega) \int_0^\tau \frac{ds}{A(s)\hat{p}_1^2(s, \omega)} ds \quad (20)$$

in order to get the complete solution of (13) as :

$$\hat{p}(\tau, t) = \alpha \hat{p}_1(\tau, \omega) + \beta \hat{p}_2(\tau, \omega) \quad (21)$$

Thus the quotient of two linearly independent solutions, required to compute the schwarzian is :

$$\frac{\hat{p}_1}{\hat{p}_2}(\tau, \omega) = \int_0^\tau \frac{ds}{A(s)\hat{p}_1^2(s, \omega)} ds \quad (22)$$

Notice that only the derivatives of this quotient are needed (see equation (15)). Thus computing the schwarzian needs only to know :

$$\left(\frac{\hat{p}_1}{\hat{p}_2} \right)_x(\tau, \omega) = \frac{1}{A(\tau)\hat{p}_1^2(\tau, \omega)} \quad (23)$$

and this can be computed recursively from the outputs A (from ζ) and p_1 of the layer-peeling algorithm.

⁴Index 1 stands for one particular solution

4 Conclusion

The direct layer-peeling algorithm presented here should be compared to the usual algorithms making based on the local reflection coefficient. This is straightforward as the algorithm uses the same measurements as input impedance measurements. Because of its well-investigated mathematical properties, one interest in using the schwarzian derivative is the possibility to analyze results coming from simulations or experiments, in conjunction with the direct layer-peeling algorithm that has been presented. One hope is the possible implications for the analysis of reconstructed bores in terms of musical characterization. But all this has to be investigated in more details, together with simulations and experiments, to prove its real efficiency and usefulness for practical bore reconstruction.

References

- [1] N. Amir, U. Shimony, and G. Rosenhouse. A discrete model for tubular acoustic systems with varying cross-section - the direct and inverse problems part 1 : Theory. *Acustica*, 81:450–462, 1995.
- [2] N. Amir, U. Shimony, and G. Rosenhouse. A discrete model for tubular acoustic systems with varying cross-section - the direct and inverse problems part 2 : Experiments. *Acustica*, 81:463–474, 1995.
- [3] A.H. Benade and M.I. Ibisí. Survey of impedance methods and a new piezo-disk-driven impedance head for air columns. *J. Acoust. Soc. Am.*, 81:1152–1167, 1987.
- [4] A.H. Benade and E.V. Jansson. On plane and spherical waves in horns with non uniform flare. *Acustica*, 31(2):79–98, 1974.
- [5] A.M. Bruckstein and T. Kailath. Inverse scattering for discrete transmission-line models. *SIAM Review*, 29(3):359–389, 1987.
- [6] A.M. Bruckstein, I. Koltracht, and T. Kailath. Inverse scattering with noisy data. *SIAM J. Sci. Stat. Comput.*, 7(4):1331–1349, 1986.
- [7] A.M. Bruckstein, B.C. Levy, and T. Kailath. Differential methods in inverse scattering. *SIAM J. Appl. Math.*, 45(2):312–334, 1985.
- [8] K.P. Bube and R. Burridge. The one-dimendional inverse problem of reflection seismology. *SIAM Review*, 25(4):497–559, 1983.
- [9] J.M. Buick, J. Kemp, D.B. Sharp, M. van Walstijn, D.M. Campbell, and R.A. Smith. Distinguishing between similar tubular objects using pulse reflectometry : a study of trumpet and cornet leadpipes. *Meas. Sci. Technol.*, 13:750–757, 2002.
- [10] A. Chaigne and J. Kergomard. *Acoustique des instruments de musique*. Belin, 2008.
- [11] J.P. Dalmont. Acoustic impedance measurement, part i : a review. *J. Sound and Vibration*, 243(3):427–439, 2001.
- [12] J.P. Dalmont, Curtit M., and A.F. Yahaya. On the accuracy of bore reconstruction from input impedance measurements : application to bassoon crook measurements. *J. Acoust. Soc. Am.*, 131(1):708–714, 2012.
- [13] E. Eisner. The complete solution of the "webster" horn equation. *J. Acoust. Soc. America*, 41:1126–1146, 1967.
- [14] I.M. Gelfand and B.M. Levitan. On the determination of a differential equation from is spectral function. *Amer. Math. Soc. Transl.*, 1:253–304, 1955.
- [15] P.M. Morse and H. Feshbach. *Methods of theoretical physics*. Mc Graw Hill, 1953.
- [16] V. Ovsienko and S. Tabachnikov. What is... the schwarzian derivative ? *Notices of the Am. Math. Soc.*, 56(1):34–36, 2009.
- [17] J.R. Resnick. Use of step response in inverse scattering. *J. Acoust. Soc. Am.*, 68(S1):S85–S86, 1980.
- [18] B.D. Sharp. *Acoustic pulse reæctometry for the measurement of musical wind instruments*. PhD thesis, Univ. Edinburgh, U.K., 1996.
- [19] J.A. Ware and K. Aki. Continuous and discrete inverse-scattering problems in a stratified elastic medium. i. plane wave at normal incidence. *J. Acoust. Soc. Amer.*, 45(4):911–921, 1969.